Appendix: Mathematical Framework for DAA Detection Timing Analysis

A.1 Introduction

This appendix presents a mathematical framework for analyzing detection delay distributions in Detect and Avoid (DAA) systems, focusing on key parameters that affect system performance and risk assessment. The framework provides a foundation for the probabilistic and statistical analysis methodology described in Section 2.2.

A.2 System Model

A.2.1 Detection Time Components

The total detection time random variable $\mathcal{T}_{det}$ is composed of several components, aligned with the ASTM DAA Timing Standard [1] described in Section 2.2.3:

$$\mathcal{T}_{det} = \mathcal{T}_{scan} + \mathcal{T}_{relay} + \mathcal{T}_{filter} + \mathcal{T}_{pub} \tag{1}$$

where:

- $\mathcal{T}_{scan}$: Random variable for sensor scanning time
- $\mathcal{T}_{relay}$: Random variable for communication relay time
- $\mathcal{T}_{filter}$: Random variable for noise filtering time
- $\mathcal{T}_{pub}$: Random variable for data publication time

A.2.2 Nominal Time Components

Let τ_x denote the nominal (expected) value of time component $\mathcal{T}_x$. The RTCA DO-365B standard [2] specifies the following nominal delay values:

A.3 Separation Distance Model [3]

A.3.1 Trajectory Analysis

For N aircraft over time interval $[0, T]$, we define:

Category	Component	Nominal Value τ_x (ms)
Scan	ADS-B	2500
	ATAR	500
	GBSS	100
	Active Surveillance	1000
Relay	Tracker	1000
	Alerting	1000
	Guidance	1000
Filter	C2 Link	1000
	GBSS Forwarding	1000
Publish	Display	500

Table 1: RTCA DO-365B Nominal Delay Components

- $\mathcal{D}_{ij}(t)$: Random variable for separation distance between aircraft i and j at time t
- d_{min}: NMAC threshold distance (constant)
- d_{max}: Maximum detection distance (constant)

The set of threshold crossing times $\mathcal{T}_{ij}$ is:

$$\mathcal{T}_{ij} = \{t \in [0, T] : \mathcal{D}_{ij}(t) \in \{d_{min}, d_{max}\}\} = \{t_{ij}^{(1)}, \ldots, t_{ij}^{(n_{ij})}\} \tag{2}$$

A.3.2 Measurement Distribution

Let Δt be the fixed detection algorithm duration. For trajectory segment k:

$$\mathcal{D}_{ij}^k = \{\mathcal{D}_{ij}(t) : t \in [t_k^{start}, t_k^{end})\} \tag{3}$$

The set of distances at sampling instant ℓ is:

$$\mathcal{M}_\ell = \{\mathcal{D}_{ij}^k(\ell\Delta t) : k \in \{1, \ldots, n_{ij}\}, (i, j) \in \{1, \ldots, N\}^2\} \tag{4}$$

The expected distance at sampling instant ℓ is:

$$\mu_\ell = \mathbb{E}[\mathcal{M}_\ell] = \frac{1}{|\mathcal{M}_\ell|} \sum_{m \in \mathcal{M}_\ell} m \tag{5}$$

A.4 Detection Probability Model

A.4.1 Success Probability Sequence

Let E be the random vector of environmental conditions. The probability of successful detection is:

$$p(d|E) = \mathbb{P}(\text{detection}|E) \tag{6}$$

The sequence of detection probabilities is:

$$p_\ell = p(\mu_\ell|E) \tag{7}$$

A.4.2 Detection Time Distribution

Let $\mathbb{A}$ be the random variable for the number of steps until detection. Then:

$$\mathbb{P}(\mathbb{A} = a) = p_a \prod_{\ell=1}^{a-1}(1 - p_\ell) \tag{8}$$

For the final time step $a_{max} = \lceil T/\Delta t \rceil$:

$$\mathbb{P}(\mathbb{A} = a_{max}) = \prod_{\ell=1}^{a_{max}-1}(1 - p_\ell) \tag{9}$$

The random detection delay is $\mathcal{T}_{det} = \mathbb{A}\Delta t$.

A.5 Environmental Factors Model

The detection time $\mathcal{T}_{det}$ is influenced by various environmental and system factors, as summarized in Table 2, which presents key factors such as field of view, aircraft speed, and environmental conditions along with their effects on detection performance.

A.5.1 Basic Probability Model

Key random variables:

- $\mathcal{T}_{det}$: Detection time

Factor	Description	Effect on Detection Time	Rationale
Field of View (FoV)	The horizontal and vertical angles through which the system can detect objects.	Decreases with wider FoV.	Wider coverage enables faster detection.
Aircraft Speed ($\mathcal{V}$)	The speed of the uncooperative aircraft (UAS).	Decreases with faster speed.	Faster objects enter detection range sooner.
Encounter Angles (θ)	The relative angle between the UAS and the uncooperative aircraft during detection.	Decreases with shallower angles.	Shallow angles provide longer exposure time.
False Positive Rate (FPR)	The number of incorrect detections produced by the system.	Increases with higher false positive rates.	System overload slows processing.
Sensor Update Rate	The rate at which the sensor updates its data.	Increases with slower update rates.	Slower updates delay detection.
Environmental Conditions ($\mathcal{E}$)	Weather conditions, visibility, smoke, lighting.	Increases in poor conditions.	Poor visibility reduces effectiveness.
Distance and Altitude ($\mathcal{D}$, $\mathcal{A}$)	The distance between the DAA system and the uncooperative aircraft.	Increases with distance	Detection difficulty scales with distance.
Aircraft Size (S)	The size of the intruding aircraft.	Decreases with larger aircraft size	Larger objects are easier to detect.
Clutter ($\mathcal{C}$)	Presence of background objects.	Increases with more clutter	More clutter causes processing delays.
Other Factors ($\mathcal{F}$)	External factors like sensor noise, hardware limitations.	Varies	Depends on specific factors.

Table 2: Environmental and System Factors Affecting Detection Time

- $\mathcal{D}$: Planar Distance to intruder
- $\mathcal{A}$: Altitude difference
- $\mathcal{V}$: Visibility conditions
- $\mathcal{C}$: Clutter
- $\mathcal{E}$: Other environmental factors

The conditional probability of detection:

$$\mathbb{P}(\mathcal{T}_{det}|\mathcal{D},\mathcal{A},\mathcal{C},\mathcal{V},\mathcal{E}) = \frac{\mathbb{P}(\mathcal{D},\mathcal{A},\mathcal{C},\mathcal{V},\mathcal{E}|\mathcal{T}_{det}) \cdot \mathbb{P}(\mathcal{T}_{det})}{\mathbb{P}(\mathcal{D},\mathcal{A},\mathcal{C},\mathcal{V},\mathcal{E})} \tag{10}$$

A.5.2 Independence Assumptions

Assuming independence:

$$\mathbb{P}(\mathcal{T}_{det}|\mathcal{D},\mathcal{A},\mathcal{C},\mathcal{V},\mathcal{E}) = \frac{\mathbb{P}(\mathcal{D}|\mathcal{T}_{det}) \cdot \mathbb{P}(\mathcal{A}|\mathcal{T}_{det}) \cdot \mathbb{P}(\mathcal{C}|\mathcal{T}_{det}) \cdot \mathbb{P}(\mathcal{V}|\mathcal{T}_{det}) \cdot \mathbb{P}(\mathcal{E}|\mathcal{T}_{det}) \cdot \mathbb{P}(\mathcal{T}_{det})}{\mathbb{P}(\mathcal{D}) \cdot \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{C}) \cdot \mathbb{P}(\mathcal{V}) \cdot \mathbb{P}(\mathcal{E})} \tag{11}$$

A.6 Probability Function Models

A.6.1 System Detection Rate

The baseline detection probability:

$$p_{base} = \mathbb{P}(\text{detection}) = 0.994 \text{ (example DAA system specification)} \tag{12}$$

A.6.2 General Logistic Function Model

For a parameter θ with nominal value θ_{nom} and scaling factor s_θ, we define a general logistic function:

$$f_\theta(\theta, \theta_{nom}, s_\theta) = \frac{1}{1 + e^{s_\theta \cdot \frac{\theta - \theta_{nom}}{\theta_{nom}}}} \tag{13}$$

The derivative of this function with respect to the scaling factor s_θ is:

$$\frac{\partial f_\theta(\theta, \theta_{nom}, s_\theta)}{\partial s_\theta} = -\frac{(\theta - \theta_{nom})}{\theta_{nom}} \cdot \frac{e^{s_\theta \cdot \frac{\theta - \theta_{nom}}{\theta_{nom}}}}{\left(1 + e^{s_\theta \cdot \frac{\theta - \theta_{nom}}{\theta_{nom}}}\right)^2} \tag{14}$$

A.6.3 Distance-Based Detection

Using the general logistic function from Equation 13, the probability of detection given distance is:

$$f_d(d) = \frac{\mathbb{P}(\mathcal{D}|\mathcal{T}_{det})}{\mathbb{P}(\mathcal{D})} = f_\theta(d, d_{nom}, s_d) \tag{15}$$

where:

- d_{nom} is the nominal detection range
- s_d is a distance scaling factor

A.6.4 Altitude Factor

For altitude difference h with maximum value h_{max}:

$$f_h(h) = \frac{\mathbb{P}(\mathcal{A}|\mathcal{T}_{det})}{\mathbb{P}(\mathcal{A})} = f_\theta(h, h_{max}/2, 10) \tag{16}$$

A.6.5 Visibility Factor

A simplified linear visibility model:

$$f_v(v) = v \tag{17}$$

A.6.6 Clutter Factor

The clutter factor decreases with increasing clutter c:

$$f_c(c) = 1 - c \tag{18}$$

A.6.7 Combined Detection Probability Model

The overall detection probability combining all factors:

$$p(d, \theta) = p_{base} \cdot f_d(d) \cdot \prod_i f_{\theta_i}(\theta_i) \tag{19}$$

where θ represents the vector of all parameters and f_{θ_i} represents the factor function for parameter θ_i.

A.7 Risk Assessment

The collision risk [4] over time period $[t_1, t_2]$ is:

$$\mathcal{R}([t_1, t_2]) = \max_{t \in [t_1, t_2]} \mathbb{P}(\text{collision at time } t) \tag{20}$$

The probability of collision at time t:

$$\mathbb{P}(\text{collision at time } t) = 1 - \int_0^{t_{cpa}} f_{\mathcal{T}_{avoid}}(\tau) d\tau \tag{21}$$

where $f_{\mathcal{T}_{avoid}}$ is the probability density function of the avoidance time random variable $\mathcal{T}_{avoid}$.

A.8 Sensitivity Analysis Framework

A.8.1 Parameter Sensitivity Model

To understand how different parameters affect the detection delay distribution, we define a sensitivity analysis framework. For a parameter θ that can take values from Θ, we examine the relationship between parameter variations and system performance metrics.

A.8.2 Detection Probability Function

Using the combined detection probability model defined in Equation 19, we analyze how variations in individual parameters affect the overall detection performance.

For specific parameter analysis, we substitute the appropriate factor functions defined in Section A.6:

- For distance analysis: use $f_d(d)$ from Equation 15
- For altitude analysis: use $f_h(h)$ from Equation 16
- For visibility analysis: use $f_v(v)$ from Equation 17
- For clutter analysis: use $f_c(c)$ from Equation 18

A.8.3 Sensitivity Metrics

For each parameter value $\theta \in \Theta$, we compute:

- $\mathcal{T}_{det}(\theta)$: Mean detection time
- $\sigma_{\mathcal{T}}(\theta)$: Standard deviation of detection time
- $P_{det}(\theta)$: Overall detection rate
- $\bar{d}_{det}(\theta)$: Average detection distance
- $P_{NMAC}(\theta)$: NMAC rate

A.8.4 Sensitivity Coefficients

The normalized sensitivity coefficient for metric M with respect to parameter θ is:

$$S_\theta^M = \frac{\theta_0}{M_0} \cdot \frac{\partial M}{\partial \theta}|_{\theta=\theta_0} \tag{22}$$

where θ_0 is the baseline parameter value and M_0 is the baseline metric value.

A.8.5 Discrete Approximation

For discrete parameter values, we approximate the sensitivity coefficient:

$$S_\theta^M \approx \frac{\theta_0}{M_0} \cdot \frac{M(\theta_0 + \Delta\theta) - M(\theta_0 - \Delta\theta)}{2\Delta\theta} \tag{23}$$

A.8.6 Monte Carlo Simulation Procedure

To implement the sensitivity analysis, we use a Monte Carlo approach:

Algorithm 1 Sensitivity Analysis Procedure

for each parameter $\theta \in \Theta$ **do**
  for each value θ_i in the parameter range **do**
    Configure system with $\theta = \theta_i$
    for $n = 1$ to N_{trials} **do**
      Generate encounter scenario
      Simulate detection process
      Record detection outcome and time
    end for
    Compute performance metrics $M(\theta_i)$
  end for
  Calculate sensitivity coefficients S_θ^M
end for

A.8.7 3D Visualization

To visualize the detection probability as a function of multiple parameters, we define a 3D surface:

$$P(x, y, z) = \{(d, h, v, p) : p = p(d, h, v)\} \tag{24}$$

where d is distance, h is altitude difference, v is visibility, and p is the resulting detection probability computed using Equation 19.

A.9 Conclusion

This framework provides a mathematical basis for analyzing detection delay distributions in DAA systems, incorporating both system parameters and environmental variables. The model can be used for risk assessment and system performance evaluation, as demonstrated in the simulation results presented in Sections 2.2.10 and 2.2.11.

References

[1] F38 Committee, "Standard specification for detect and avoid system performance requirements," tech. rep., ASTM International, West Conshohocken, PA, 2023.

[2] C. Serres, B. Gill, P. Reheis, and M. Edwards, "RTCA detect and avoid phase 2: Safety risk management modeling and simulation final report," *MIT Lincoln Laboratories*, 2022.

[3] G. May, “A method for predicting the number of near mid-air collisions in a defined airspace,” *Operational Research Quarterly (1970-1977)*, vol. 22, no. 3, pp. 237–251, 1971.

[4] L. R. Sahawneh and R. W. Beard, “A probabilistic framework for unmanned aircraft systems collision detection and risk estimation,” in *53rd IEEE Conference on Decision and Control*, IEEE, Dec. 2014.